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DIFFRACTION OF PULSES BY CONDUCTING WEDGES AND CONES

by

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TABLE OF CONTENTS

	Page
Abstract	i
I. Introduction	1
II. The Conducting Wedge	2
III. The Conducting Cone	7
References	15

ABSTRACT

The diffraction of a plane pulse by a conducting wedge or cone is obtained by first finding the solution due to an incident plane wave and then integrating over the propagation constant of this wave. The integrals which appear may all be evaluated. For the wedge the solution agrees with that previously obtained by Keller and Blank. For the cone, the solution is given as an infinite series in the diffraction region. Outside this region the series solution may be summed to give the expected discontinuous solution.

I. Introduction

The time-harmonic problem of diffraction of monochromatic plane waves by conducting wedges or cones was solved by Macdonald[1]. More recently, the diffraction of a plane pulse by conducting wedges was treated by Keller and Blank[2]. They used the method of conical flow, appealing to the fact that the solution is "conical" in x,y,t space, and thereby is independent of "radial" distance in a system of spherical coordinates. In this method the boundary value problem can be reduced to that of solving Laplace's equation, which can be done by conformal mapping. The results are particularly simple since they depend only on elementary functions. The same result was obtained by Kay[3] using the method of separation of variables directly on the wave equation by adopting a set of non-orthogonal coordinates in which the characteristic cone is a coordinate surface.

For the similar problem of diffraction of pulses by cones these methods are not so simply applied. To obtain the solution in this case, the method of Fourier synthesis, which applies equally to the problem of the wedge and the cone, was used. An incident plane pulse may be represented as a superposition of plane monochromatic waves of all frequencies, with appropriate weighting factors. The solution of the harmonic diffraction problem is known. Then a superposition of harmonic solutions, using the same weighting factors as in the incident pulse, will give the solution of the pulse diffraction problem. This may be proved by an appeal to the linearity of the system. The integrals which appear may all be evaluated. Within the diffraction region the solution can be expressed only as an infinite series; however, outside this region the series may be summed to find the expected discontinuous solution.

The inverse procedure, whereby the classical harmonic solution for a wedge is obtained from the pulse solution, is given for the wedge by Keller and Blank.

In Section II the method is applied to a conducting wedge, and the Keller-Blank solution is obtained. In Section III the cone problem is treated.

II. The Conducting Wedge

This problem is the same as that treated by Keller and Blank. A plane pulse, described by a field strength having the value unity behind the wavefront and zero ahead, is incident upon a conducting wedge of semiangle α. Only the case of normal incidence will be treated, since the analysis becomes very cluttered for arbitrary angles of incidence and this section is intended for illustrative purposes. The boundary condition that the wave function u vanishes upon the wedge will be taken: this condition corresponds to an electromagnetic pulse with the electric vector parallel to the generators of the wedge. The problem is then mathematically formulated as follows:

We are given the wave equation

$$\text{(1)} \qquad \text{a)} \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = 0$$

With the boundary conditions on the wedge

$$\text{b)} \qquad u = 0 \text{ for } \theta = \alpha,\ 2\pi - \alpha$$

and the initial conditions

$$\text{c)} \qquad u_{inc} = \begin{matrix} 1, & ct > x \\ 0, & ct < x . \end{matrix}$$

Thus the incident pulse reaches the origin at time $t = 0$. When $t < 0$, the configuration is illustrated in Figure 1.

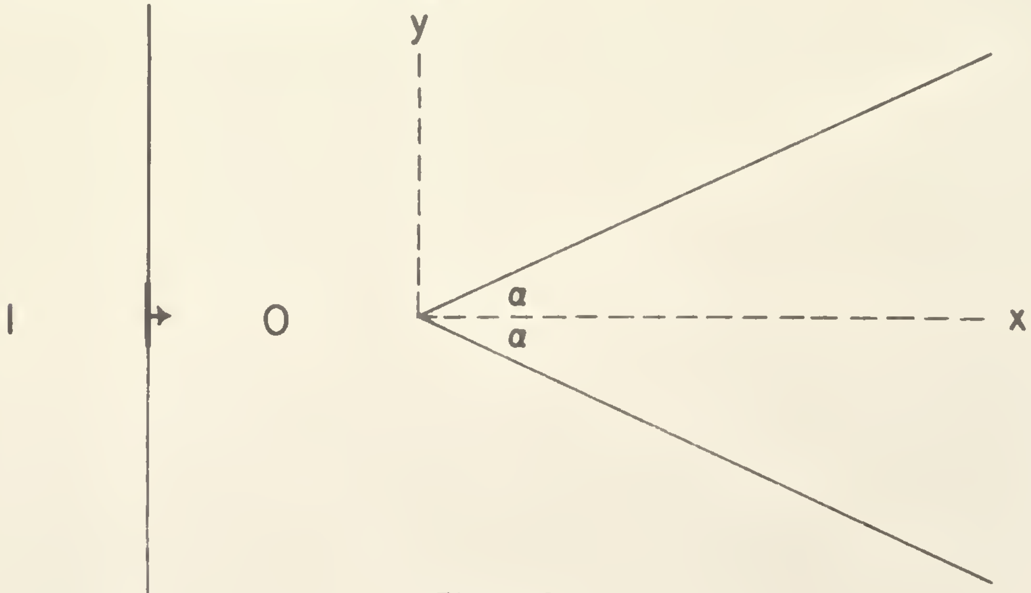

Figure 1.

Keller and Blank now apply the conical flow method. However, in this paper the relation between pulse and harmonic solutions is to be employed. The incident pulse u_{inc} may be represented by the contour integral,

$$u_{inc} = -\frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{dk}{k}e^{-ik(ct-x)}, \qquad a > 0. \tag{2}$$

To see the truth of this, we observe that the integral has a pole at the origin. If $ct > x$, the path of integration may be closed in the lower half plane, and the residue at the pole is unity. If $ct < x$, the path may be closed in the upper half plane, so that the pole is outside the contour and the integral vanishes.

The harmonic problem is now defined by asking for that solution of the wave equation (1a) and boundary conditions (1b) which corresponds to the incident wave $\exp -ik(ct-x)$. Let this solution be denoted by $u(x,y,k)\exp -ikct$. A solution of the pulse problem is now given by:

$$u(x,y,t) = -\frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{dk}{k}u(x,y,k)e^{-ikct}. \tag{3}$$

It is clear that (3) is a solution of the wave equation and the boundary conditions. The function $u(x,y,k)$ is the sum of an incoming plane wave and outgoing waves. For negative values of t, the path of integration may be closed in the upper half plane for the outgoing wave portion of $u(x,y,k)$, and the integral will vanish, since it is assumed that $u(x,y,k)$ has no singularities for $\operatorname{Im} k > 0$. The plane wave part of $u(x,y,k)$ gives rise to the incident pulse. Consequently, the function defined by (3) actually satisfies the conditions of the problem.

The harmonic solution was found by Macdonald by using an analogy with the electrostatic case. However, that method does not prove to be simple for the cone problem. Accordingly, we shall derive Macdonald's solution by a somewhat different method which applies equally well to the cone. The clearest method replaces the incident wave problem by the corresponding problem for a point source on the negative x-axis; we find the solution of the latter problem, and then let the source recede to minus infinity. This latter problem may be written as an inhomogeneous wave equation problem, as has been pointed out by Schwinger and Levine[4]. Thus we wish to solve:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+k^2\right]V(r,\theta,r') = -\delta(r-r')\delta(\theta-\pi)/r, \tag{4}$$

where $V = 0$ for $\theta = \alpha,\ 2\pi-\alpha$.

The functions $\sin\left[n\pi(\theta-\alpha)/2(\pi-\alpha)\right]$ form an orthogonal and complete set and vanish for $\theta = \alpha,\ 2\pi-\alpha$. The normalization constant of this set is $(\pi-\alpha)^{-1/2}$. The solution may then be expanded in terms of this set of functions. Let $\lambda = \pi/2(\pi-\alpha)$. Then we assume the expansion

$$V = \sum_{1}^{\infty} V_n(r,r')\sin n\lambda(\theta-\alpha)\sin n\lambda(\pi-\alpha) \tag{5}$$

$$\sum^{\infty} V_n(r,r')\sin n\lambda(\theta-$$

When the expansion (5) is inserted into equation (4) and use is made of the orthogonality property, the following equation results for V_n:

$$(6)\qquad \left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + k^2 - \frac{n^2\lambda^2}{r^2} \right] V_n(r,r') = -\delta(r-r')/(\pi-\alpha)r.$$

The function V_n satisfies the homogeneous form of equation (6) except for $r = r'$. We wish to find that solution which is regular at the origin and behaves like an outgoing wave at infinity. Since Im $k > 0$, the appropriate solution is given by:

$$(7)\qquad \begin{aligned} V_n &= AJ_{n\lambda}(kr)H^{(1)}_{n\lambda}(kr') && r < r' \\ &= AJ_{n\lambda}(kr')H^{(1)}_{n\lambda}(kr) && r > r'. \end{aligned}$$

The differential equation (6) states that $r\, d/dr\, V_n(r,r')$ has the jump $-1/(\pi-\alpha)$ at $r = r'$. When the solution (7) is inserted into this, the constant A is determined to be $i\lambda$. Thus the solution of (4) for $r < r'$ is given by:

$$(8)\qquad V = i\lambda \sum_1^\infty J_{n\lambda}(kr)H^1_{n\lambda}(kr')\sin n\lambda(\theta-\alpha)\sin n\pi/2 \quad .$$

Now the source is to approach minus infinity, which means r' approaches infinity. Upon substituting the asymptotic form for the Hankel function, there results:

$$(9)\qquad V \sim \sqrt{\frac{2}{\pi kr'}}\; e^{i(kr'-\pi/4)} i\lambda \sum_1^\infty J_{n\lambda}(kr)e^{-in\lambda\pi/2}\sin n\lambda(\theta-\alpha)\sin n\pi/2.$$

The solution in the absence of the wedge is given by

$$i/4\; H^1_0\left[k\left\{r^2+r'^2-2rr'\cos(\theta-\pi)\right\}^{1/2}\right],$$

and the asymptotic form of this for $r' \to \infty$ is $i/4\sqrt{2/\pi kr'}\;\, e^{i(kr'-\pi/4)}e^{-ikr\cos\theta}$ Consequently, the solution of the incident plane wave problem is given by:

$$(10)\qquad u(x,y,k) = 4\lambda \sum_1^\infty J_{n\lambda}(kr)e^{-in\lambda\pi/2}\sin n\lambda(\theta-\alpha)\sin n\pi/2.$$

This is the same as Macdonald's solution.

The harmonic solution (10) is to be inserted into the integral (3) to obtain the pulse solution. The only k dependence of u(x,y,k) is through the Bessel functions. Assuming that the interchange of summation and integration is permissible, the pulse solution u(x,y,t) is given by the series:

$$(11)\qquad u(x,y,t) = 4\lambda \sum_1^\infty e^{-in\lambda\pi/2}\sin n\lambda(\theta-\alpha)\sin \frac{n\pi}{2} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{dk}{k} J_{n\lambda}(kr)e^{-ikct} \quad .$$

The integral appearing here is to be evaluated. Since $n\lambda$ is positive, the integrand of (11) has an integrable singularity at $k = 0$. Accordingly, the path of integration may be deformed into the real axis with a loop above the origin. On the negative real axis, the phase of k is π. Since

$$J_{n\lambda}(kre^{i\pi}) = e^{in\lambda\pi} J_{n\lambda}(kr) \text{ ,}$$

the integral is equal to:

$$(12) \quad -\frac{1}{2\pi i} \int_{ia-\infty}^{ia+\infty} \frac{dk}{k} J_{n\lambda}(kr)e^{-ikct} = -\frac{1}{2\pi i} \int_{o}^{\infty} \frac{dk}{k} J_{n\lambda}(kr)\left[e^{-ikct} - e^{in\lambda\pi}e^{ikct}\right]$$

$$= -\frac{1}{2\pi i} \int_{o}^{\infty} \frac{dk}{k} J_{n\lambda}(kr)\left[(1-e^{in\lambda\pi})\cos kct - i(1+e^{in\lambda\pi})\sin kct\right] .$$

These integrals are special cases of the Weber-Schafheitlin integral, given in detail by Watson[5]. The result depends upon the relative value of r and ct. When $r < ct$, we obtain

$$(13) \quad -\frac{1}{2\pi i} \frac{r^{n\lambda}}{n\lambda\left[ct+\sqrt{c^2t^2-r^2}\right]^{n\lambda}} \left[(1-\exp in\lambda\pi)\cos \frac{n\lambda\pi}{2} - i(1+\exp in\lambda\pi)\sin n\lambda\pi/2\right]$$

$$= \frac{1}{\pi} \frac{r^{n\lambda}}{n\lambda\left[ct+\sqrt{c^2t^2-r^2}\right]^{n\lambda}} e^{in\lambda\pi/2} \sin n\lambda\pi$$

On the other hand, if $r > ct$, the integral takes the form:

$$(14) \quad -\frac{1}{2\pi i} \frac{1}{n\lambda} \left[(1-\exp in\lambda\pi)\cos\left\{n\lambda\sin^{-1}ct/r\right\} - i(1+\exp in\lambda\pi)\sin\left\{n\lambda\sin^{-1}ct/r\right\}\right]$$

$$= \frac{1}{\pi} \frac{1}{n\lambda} \exp(in\lambda\pi/2)\sin\left[n\lambda\left\{\frac{\pi}{2} + \sin^{-1}ct/r\right\}\right]$$

$$= \frac{1}{\pi} \frac{1}{n\lambda} \exp(in\lambda\pi/2)\sin\left[n\lambda\cos^{-1}(-ct/r)\right].$$

Following Keller and Blank, the variable ρ is introduced, defined by:

$$(15) \quad \rho = r\left[ct + \sqrt{c^2t^2-r^2}\right] \qquad ct > r,$$

and we also define an angle β by the relation

$$(16) \quad \beta = \cos^{-1}(-ct/r) \qquad \pi/2 < \beta < \pi \qquad ct < r.$$

The solution to the pulse problem is now given by

$$(17) \quad u(x,y,t) = 4/\pi \sum_{1}^{\infty} \rho^{n\lambda} \sin \frac{n\pi}{2} \sin n\lambda\pi \sin n\lambda(\theta-\alpha)/n \qquad 0 < r < ct$$

$$= 4/\pi \sum_{1}^{\infty} \sin \frac{n\pi}{2} \sin n\lambda\beta \sin n\lambda(\theta-\alpha)/n \qquad r > ct.$$

To find the sum of these series, we use an identity derived from a formula of Dwight[6],

$$(18)\qquad \sum_{1}^{\infty} \frac{x^n}{n} \sin ny = \tan^{-1} \frac{x\sin y}{1-x\cos y}\ ; \quad -\pi/2 < \tan^{-1} < \pi/2 .$$

We first apply this to the representation for $r < ct$. The product of three sines may be broken into the sum of four sines. The formula (18) may be transformed into the more useful form:

$$(19)\qquad \sum_{1}^{\infty} \frac{x^n}{n} \sin \frac{n\pi}{2} \cos ny = 1/2 \tan^{-1}\left\{2x\cos y/(1-x^2)\right\}; \quad -\pi/2 < \tan^{-1} < \pi/2$$

which is obtained by writing $\pi/2 \pm y$ for y in (18) and adding the results. Thus we obtain:

$$(20)\qquad u(x,y,t) = 1/\pi \left[\tan^{-1} \frac{2\rho^{\lambda}\cos\lambda(\pi-\theta+\alpha)}{1-\rho^{2\lambda}} - \tan^{-1} \frac{2\rho^{\lambda}\cos\lambda(\pi+\theta-\alpha)}{1-\rho^{2\lambda}}\right]; \ ct > r.$$

By combining the arctangents this formula may be reduced to the formula (16) of Keller and Blank, recalling that (20) holds for nose-on incidence ($\psi=0$ in the notation of Keller and Blank).

Outside the diffraction region, we must use the second equation of (17). This implies taking the limit as x approaches 1 in (19). The limit of (19) is $\pi/4$ if $\cos y$ is positive and $-\pi/4$ if $\cos y$ is negative. The values of y are $\lambda(\beta-\theta+\alpha)$ and $\lambda(\beta+\theta-\alpha)$. Thus the solution is given by:

$$(21)\qquad u(x,y,t) = \lim_{x\to 1} 2/\pi \left[\tan^{-1} \frac{2x\cos\lambda(\beta-\theta+\alpha)}{1-x^2} - \tan^{-1} \frac{2x\cos\lambda(\beta+\theta-\alpha)}{1-x^2}\right]$$

We now draw Figure 2, which shows where the cosines change sign.

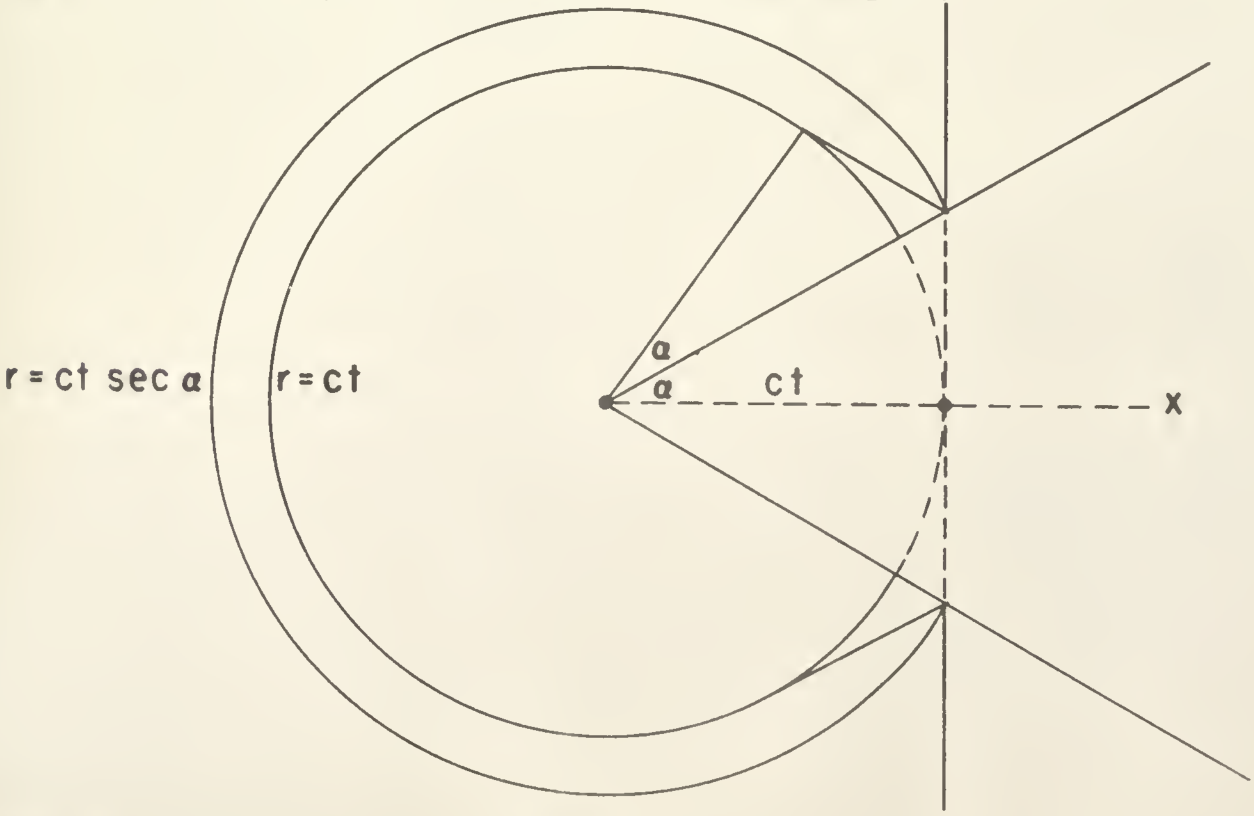

Figure 2.

Upon the upper vertical line, $\lambda(\beta+\theta-\alpha)=\pi/2$, while on the lower vertical line $\lambda(\beta-\theta+\alpha)=-\pi/2$. Outside the circle $r = ct \sec\alpha$, β lies between $\pi/2$ and $\pi-\alpha$. Thus, between the upper vertical line and the wedge, both cosines are negative, so the limit is zero. Similarly, between the lower vertical line and the wedge, both cosines are positive and the limit is zero. In the region outside the circle $r = ct \sec\alpha$ to the left of the vertical line, $\cos\lambda(\beta+\theta-\alpha)$ is positive, while $\cos\lambda(\beta-\theta+\alpha)$ is negative, so the limit is unity.

Upon the circle $r = ct \sec\alpha$, $\beta = \pi-\alpha$, and the cosines have opposite signs throughout. Between the circles $r = ct$ and $r = ct\sec\alpha$, β lies between $\pi-\alpha$ and π. The upper slanted line has the equation $r\cos(2\alpha-\theta)=ct$, whence $\lambda(\beta-\theta+\alpha)=\pi/2$ along this line. Along the lower slanted line, $\beta=3\pi-2\alpha-\theta$, and $\lambda(\beta+\theta-\alpha)=3\pi/2$. Thus, between the upper slanted line and the wedge, both cosines are negative, and the limit is zero. Between the lower slanted line and the wedge, both cosines are positive, and the limit is zero. In the region between the circles but to the left of the slanted lines, $\cos\lambda(\beta-\theta+\alpha)$ is positive, while $\cos\lambda(\beta+\theta-\alpha)$ is negative, so the limit is unity.

Consequently, the geometrical optics pulse solution, which holds outside the diffraction region, has been obtained by transforming the harmonic solution. It is given by superposing the incident pulse and reflected pulse, and setting the field equal to zero in the region between the reflected pulse front and the wedge. This solution was obtained by Keller and Blank directly, by appealing to the known manner of propagation of discontinuities.

III. The Conducting Cone

The geometrical picture for the conducting case is the same as Figure 1 for the wedge, except that the figure is to be viewed as axially symmetric rather than being two-dimensional. We introduce a system of spherical coordinates r, θ, ϕ with the pole in the direction of the positive x-axis. The equation of the cone is given by $\theta = \alpha$. For nose-on incidence, the system possesses rotational symmetry, and there is no ϕ dependence.

The solution goes through the same sequence as before. First the point source problem is solved. The limit of this solution as the source moves to infinity gives the harmonic solution. Substitution of the harmonic solution in the integral (3) gives the pulse solution. Thus, we first wish to solve:

(22) $$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + k^2\right]V(r,\theta,r') = -\delta(r-r')\delta(\theta-\pi)/2\pi r^2\sin\theta;$$

$$V = 0 \text{ on the cone; } \theta = \alpha.$$

To solve this, we introduce the eigenfunctions in the θ-direction. A set of numbers ν are defined by the equation $P_\nu(-\cos\alpha)=0$, where P_ν refers to the Legendre function of order ν. In general ν will be real but not integral. The functions $P_\nu(-\cos\theta)$ are orthogonal over the range $\alpha \leq \theta \leq \pi$, and they take on the value 1 at $\theta = \pi$. These functions constitute a complete set, and they may be used for expanding the function V. Thus we write:

(23) $$V(r,\theta,r') = \sum V_\nu(r,r')\, P_\nu(-\cos\theta)/N_\nu$$

$$N_\nu = \int_\alpha^\pi P_\nu^2(-\cos\theta)\sin\theta d\theta$$

$$= -\frac{1}{2\nu+1}\sin^2\alpha \frac{\partial}{\partial(-\cos\alpha)} P_\nu(-\cos\alpha)\frac{\partial}{\partial\nu} P_\nu(-\cos\alpha).$$

The value of N_ν, which is the normalization constant of the functions P_ν, is obtained by using the integral

(24) $$\int_\alpha^\pi P_\nu(-\cos\theta)P_\mu(-\cos\theta)\sin\theta d\theta =$$

$$\sin^2\alpha \; \frac{P_\nu(-\cos\alpha)\frac{\partial}{\partial(-\cos\alpha)}P_\mu(-\cos\alpha) - P_\mu(-\cos\alpha)\frac{\partial}{\partial(-\cos\alpha)}P_\nu(-\cos\alpha)}{\mu(\mu+1) - \nu(\nu+1)} .$$

This integral, which is readily derived from the differential equations satisfied by P_μ and P_ν, holds for all values of μ and ν. Letting ν be a root of $P_\nu(-\cos\alpha)=0$ cancels the first term in the numerator. Taking the limit of the ratio as μ approaches ν then yields the formula for N_ν given above.

Upon multiplying the expansion (23) by $\sin\theta P_\nu(-\cos\theta)$, integrating from α to π, and using the orthogonality properties of the functions, the following equation for V_ν results:

(25) $$\left[\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} + k^2 - \frac{\nu(\nu+1)}{r^2}\right] V_\nu(r,r') = -\delta(r-r')/2\pi r^2 .$$

Since the boundary conditions on the cone are already satisfied by the separate terms of the expansion (23), the boundary conditions on V_ν are that it be regular at the origin and represent outgoing waves at infinity. The solutions of the homogeneous form of equation (25) are spherical Bessel functions. Thus we may write:

$$(26)\qquad V_\nu(r,r') = A\,\frac{J_{\nu+1/2}(kr)}{r^{1/2}}\;\frac{H^{(1)}_{\nu+1/2}(kr')}{r'^{1/2}}\qquad r<r'$$

$$= A\,\frac{J_{\nu+1/2}(kr')}{r'^{1/2}}\;\frac{H^{(1)}_{\nu+1/2}(kr)}{r^{1/2}}\qquad r>r'.$$

The differential equation (25) states that $r^2 dV_\nu/dr$ has the jump $-1/2\pi$ at $r = r'$, from which the constant A is determined to be $i/4$. Consequently, the function $V(r,\theta,r')$ which satisfies (22) is given for $r < r'$ by:

$$(27)\qquad V(r,\theta,r') = i/4\sum_\nu \frac{J_{\nu+1/2}(kr)}{r^{1/2}}\;\frac{H^{(1)}_{\nu+1/2}(kr')}{r'^{1/2}}\;\frac{P_\nu(-\cos\theta)}{N_\nu}\quad .$$

The next step is to find the asymptotic form of this expression as r' approaches infinity. Upon substituting the asymptotic form for the Hankel-functions, we obtain:

$$(28)\ V(r,\theta,r')\to\left(\frac{2}{\pi k}\right)^{1/2}\frac{e^{i(kr'-\pi/4)}}{r'}\;\frac{i}{4}\sum_\nu \frac{J_{\nu+1/2}(kr)}{r^{1/2}}\;e^{-i(\nu+1/2)\pi/2}\;\frac{P_\nu(-\cos\theta)}{N_\nu}\quad .$$

The solution in the absence of the cone is $e^{ik|r-r'|}/4\pi|r-r'|$, whose asymptotic form as r' approaches infinity is $e^{ikr'+ikx}/4\pi r'$. Consequently, the harmonic solution $u(\vec{1},\theta,k)$ is given by:

$$(29)\qquad u(r,\theta,k) = (2\pi)^{1/2}\sum_\nu \frac{J_{\nu+1/2}(kr)}{(kr)^{1/2}}\;e^{-i\nu\pi/2}P_\nu(-\cos\theta)/N_\nu .$$

This solution is very similar in structure to the wedge solution (10). Here k appears only in the spherical Bessel function, while for the wedge k only appeared in the cylindrical Bessel function. The integration over k this time will lead to Legendre functions.

The harmonic solution (29) is to be substituted into the integral (3) to obtain the pulse solution. Assuming that summation and integration may be interchanged, the pulse solution is given by :

$$(30)\ u(r,\theta,t)=(2\pi)^{1/2}\sum_\nu e^{-i\nu\pi/2}P_\nu(-\cos\theta)/N_\nu\,\frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{dk}{k}\,\frac{J_{\nu+1/2}(kr)}{(kr)^{1/2}}\;e^{-ikct}\quad .$$

The integral appearing here may be evaluated by straightforward but laborious transformations. First consider the case $r < ct$. Since $J_{\nu+1/2}(kr) = O(e^{|k|r})$ for large k, in this case the integration path may be closed in the lower half-plane, since it is exponentially small there because $\exp -ik(ct-r)$ is exponentially small. The resulting path may be deformed into a loop around the negative imaginary axis since the

large quartercircles will not contribute. Now the integrand has a branch point of order ν at the origin. Writing arg k = $-\pi/2$ on the right side of the integration path, and arg k = $3\pi/2$ on the left side, the integral becomes:

$$(31) \qquad I = -\frac{1}{2\pi i}\int_0^\infty \frac{dk}{k}\, e^{-kct}\left[\frac{J_{\nu+1/2}(kre^{-i\pi/2})}{(kre^{-i\pi/2})^{1/2}} - \frac{J_{\nu+1/2}(kre^{i3\pi/2})}{(kre^{i3\pi/2})^{1/2}}\right].$$

These Bessel functions of pure imaginary argument may be expressed in terms of the modified Bessel function. The result of this transformation is:

$$I = e^{i\nu\pi/2}\,\frac{\sin\nu\pi}{\pi}\int_0^\infty \frac{dk}{k}\, e^{-kct}\,\frac{I_{\nu+1/2}(kr)}{(kr)^{1/2}} \quad .$$

This integral is a special case of one given in Watson's[7] treatise. Thus it has the value:

$$(32) \qquad I = e^{i\nu\pi/2}\,\frac{\sin\nu\pi}{\pi}\;\frac{\cos(\nu+1/2)\pi}{\sin\nu\pi}\;\frac{\sinh\beta Q_\nu^{-1}(\cosh\beta)}{(\pi/2)^{1/2}}$$

$$= -\,(2/\pi)^{1/2}e^{i\nu\pi/2}\,\frac{\sin\nu\pi}{\pi}\;\sinh\beta Q_\nu^{-1}(\cosh\beta),$$

where

$$\cosh\beta = ct/r > 1.$$

Here Q_ν^{-1} denotes the associated Legendre function of the second kind. Thus the solution in the diffraction region $r < ct$ is given by:

$$(33) \quad u(r,\theta,t) = -2/\pi\sum \sin\nu\pi\sinh\beta Q_\nu^{-1}(\cosh\beta)P_\nu(-\cos\theta)/N_\nu \qquad \cosh\beta = ct/r > 1.$$

In this form, the solution is useful near the tip of the cone. To see this, suppose r is small, so ct/r is much greater than unity. The asymptotic form for the Q_ν^{-1} function is given by[8]:

$$(34) \quad \sinh\beta Q_\nu^{-1}(\cosh\beta) \rightarrow -2^{\nu-1}\sqrt{\pi}\left[\Gamma(\nu)/\Gamma(\nu+3/2)\right](ct/r)^{-\nu}$$

$$\rightarrow -\left\{\sqrt{\pi}\;\Gamma(\nu)/2\Gamma(\nu+3/2)\right\}(r/2ct)^{\nu}\,.$$

Now all the values of ν are real and positive. Therefore all the terms of the series (33) tend to zero as r tends to zero for a fixed positive value of ct. In particular, the strength of the singularity at the tip is given by the smallest root of $P_\nu(-\cos\alpha)=$ If α is small, corresponding to a sharply pointed tip, the roots of this equation are given approximately by the relation[9]:

$$(35) \qquad \nu = n + \frac{1}{2\log 2/\alpha}\;,\; n = 0,1,2,\ldots$$

Thus the singularity is given by $\nu = \frac{1}{\log 2/\alpha}$ for α small. If $\alpha > \pi/6$, a fair approximation to the roots is given by :

$$(36)\qquad \nu = (n+3/4)\pi/\pi-\alpha - 1/2 \qquad\qquad n = 0,1,2,\cdots$$

Although the formula (33) represents the pulse solution throughout the diffraction region, there appears to be no way to find the sum except by numerical procedures. Thus, we shall next consider the solution in the "geometrical" region $r > ct$. Referring again to Figure 2, we shall prove that the solution is zero to the right of the incident pulse, and is unity to the left of both the incident and reflected pulses. The solution in the region between the reflected pulse and the wedge will be presented as a series similar to (33).

The contour transformation which led from (29) to (32) is only possible when $r < ct$. When $r > ct$, the method used in Section II must be employed. However, the integral that results is a hypergeometric function, and some transformations are required to express it as a Legendre function. First, the contour is deformed into the real axis with a loop above the origin, so that the phase of k is π on the negative real axis. The integral then becomes:

$$(37)\quad I=1/\pi e^{i\nu\pi/2}\int_0^\infty \frac{dk}{k}\frac{J_{\nu+1/2}(kr)}{(kr)^{1/2}}\left[\sin\frac{\nu\pi}{2}\cos kct + \cos\frac{\nu\pi}{2}\sin kct\right]$$

$$= 1/\pi e^{i\nu\pi/2}(\pi ct/2r)^{1/2}\int_0^\infty \frac{dk}{k} J_{\nu+1/2}(kr)\left[J_{-1/2}(kct)\sin\frac{\nu\pi}{2} + J_{1/2}(kct)\cos\frac{\nu\pi}{2}\right].$$

This integral is again a special case of the Weber-Schafheitlin integral referred to in Reference 5. This time we obtain the result[10]:

$$(38)\quad I = \frac{e^{i\nu\pi/2}}{\pi/2}\left[\frac{\Gamma(\nu/2)\sin 2\pi/2}{2\Gamma[(\nu+3)/2]}\ {}_2F_1\left(\nu/2,-(\nu+1)/2,1/2,(ct/r)^2\right)\right.$$
$$\left. + \frac{\Gamma[(\nu+1)/2]\cos\nu\pi/2}{\Gamma[(\nu+2)/2]}\ \frac{ct}{r}\ {}_2F_1\left(-\nu/2,(\nu+1)/2,3/2,(ct/r)^2\right)\right] .$$

using the standard notation for the hypergeometric function. This combination of hypergeometric functions can be expressed as a Legendre function. We have the formula[11]

$$(39)\qquad P_\nu^{-1}(-x)= \frac{\Gamma(\nu/2)\sin\nu\pi/2}{2\sqrt{\pi}\,\Gamma[(\nu+3)/2]}\ (1-x^2)^{-1/2}\ {}_2F_1(\nu/2,-(\nu+1)/2,1/2,x^2\ )$$
$$+ \frac{\Gamma[(\nu+1)/2]\cos\nu\pi/2}{\sqrt{\pi}\,\Gamma[(\nu+2)/2]}\ x(1-x^2)^{-1/2}\ {}_2F_1(-\nu/2,(\nu+1)/2,3/2,x^2\ )$$

whence I is evaluated as:

$$(40)\qquad I =\left(e^{i\nu\pi/2}/\sqrt{2\pi}\right)\sin\beta\ P_\nu^{-1}(-\cos\beta) \qquad\qquad \cos\beta = ct/r .$$

Consequently, the result for $u(r,\theta,t)$ in the "geometrical" region is:

$$(41)\quad u(r,\theta,t) = \sum \sin\beta P^{-1}(-\cos\beta)P(-\cos\theta)/N \qquad\qquad \cos\beta = ct/r < 1$$

The next step is to find the sum of this series wherever such a process is possible. First, suppose $r > ct \sec \alpha$. Then $\cos\beta < \cos\alpha$, so β is an angle between α and $\pi/2$. In view of the orthogonality of the functions $P_\nu(-\cos\theta)$, the coefficient of $P_\nu(-\cos\theta)/N_\nu$ in the expansion of $u(r,\theta,t)$ is $\int_\alpha^\pi u(r,\theta,t)P_\nu(-\cos\theta)\sin\theta d\theta$. Assume that $u = 0$ for $\alpha < \theta < \beta$, and $u = 1$ for $\beta < \theta < \pi$. The vertical line of Figure 2 corresponds to $\theta=\beta$, since on it $r\cos\theta = ct$. The integral may be evaluated by a standard formula[12], and is just $\sin\beta P_\nu^{-1}(-\cos\beta)$. Therefore, the series (41) is just equal to the "geometrical" pulse solution outside this outer sphere.

The solution between the spheres $r = ct$ and $r = ct\sec\alpha$ is much more difficult to obtain and requires a rather elaborate transformation of contour integrals. We first use the Wronskian relationship

$$P_\nu(-x) \frac{d}{dx} P_\nu(x) - P_\nu(x) \frac{d}{dx} P_\nu(-x) = \frac{2\sin\nu\pi}{\pi(1-x^2)} \tag{42}$$

to express the normalization constant N_ν in the form:

$$N_\nu = - \frac{1}{2\nu+1} \; \frac{2\sin\nu\pi}{\pi} \; \frac{\frac{\partial}{\partial\nu} P_\nu(-\cos\alpha)}{P_\nu(\cos\alpha)} \quad . \tag{43}$$

Therefore, $u(r,\theta,t)$ may be expressed in the form:

$$u(r,\theta,t) = -\sum(2\nu+1) \frac{\pi}{2\sin\nu\pi} \frac{P_\nu(\cos\alpha)}{\frac{\partial}{\partial\nu} P_\nu(-\cos\alpha)} \sin\beta P_\nu^{-1}(-\cos\beta)P_\nu(-\cos\theta). \tag{44}$$

We recall that this sum goes over the positive roots of $P_\nu(-\cos\alpha) = 0$. Now $\frac{\partial}{\partial\nu} \frac{1}{P_\nu(-\cos\alpha)}$ is the residue of $1/P_\nu(-\cos\alpha)$, regarding ν as a general complex variable. Consequently, the sum in (44) may be replaced by a contour integral:

$$u(r,\theta,t) = - \frac{1}{2\pi i} \oint_C d\nu(2\nu+1)\frac{\pi}{2\sin\nu\pi} \frac{P_\nu(\cos\alpha)}{P_\nu(-\cos\alpha)} \sin\beta P_\nu^{-1}(-\cos\beta)P_\nu(-\cos\theta) \tag{45}$$

in which the contour C is composed of small circles around the zeros of $P_\nu(-\cos\alpha)$. The only other singularities of the integrand are at the zeros of $\sin\nu\pi$. If circles at these poles are added and subtracted from the contour, the added part may be combined with the contour C to replace it by a loop around the positive real axis. Evaluating the residues at the subtracted part gives the series:

$$\sum_0^\infty (n+\tfrac{1}{2}) \sin\beta P_n^{-1}(\cos\beta)P_n(\cos\theta). \tag{46}$$

Since $ct < r < ct\sec\alpha$, $0 < \beta < \alpha$, while $\alpha < \theta < \pi$. The series (46) may be shown immediately to be equal to unity under these conditions. Thus, we have obtained the integral representation for μ:

$$(47)\quad u(r,\theta,t)= 1-\frac{1}{2\pi i}\int_{\infty}^{0+} d\nu(2\nu+1)\,\frac{\pi}{2\sin\nu\pi}\,\frac{P_\nu(\cos\alpha)}{P_\nu(-\cos\alpha)}\,\sin\beta P_\nu^{-1}(-\cos\beta)P_\nu(-\cos\theta),$$

in which the phase of $-\nu$ goes from $-\pi$ to π as ν traverses the contour.

The next step is to reform this contour into the vertical line $x=-\frac{1}{2}+iy$, $-\infty < y < \infty$. The integrand has no singularities in the complex plane off the real axis, so that the contour C may be replaced by the vertical line plus two connecting quarter-circles. Now, since $\sin\nu\pi = -\cos(\nu+1/2)\pi$, and $P_\nu(x)$ is an even function of $\nu+1/2$, all the factors in the integrand are even functions of y except $2\nu+1$, which is odd. Therefore, the integral along the vertical line vanishes. If we can show that there is no contribution from the quarter-circles, the integral in (47) will be zero, and $u(r,\theta,t)$ will equal unity. We shall find the conditions under which the quarter-circles do not contribute.

For values of $|\nu|$ which are large in comparison to unity, we have the asymptotic formula[13]:

$$(48)\qquad P_\nu^\mu(\cos\theta)=\frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}\,\frac{\cos\left[(\nu+1/2)\theta-\frac{\pi}{4}+\mu\pi/2\right]}{\sqrt{2\sin\theta}}$$

When $\mathrm{Im}\nu$ is large, as it is on the quarter-circles, the cos may be replaced by that exponential which is large. Applying formula (48) to the integrand in (47), and using further asymptotic formulas for the Gamma-functions gives for the integrand the following asymptotic value:

$$(49)\qquad A(\nu)\exp -|\mathrm{Im}\nu|\,(\theta-2\alpha+\beta),$$

where $A(\nu)$ denotes an algebraic function of ν. This expression will be exponentially small if θ is greater than $2\alpha-\beta$. Now the equation of the slanting lines of Figure 2 is $r = ct\cos(2\alpha-\theta)$, whence on the slanting lines $\theta = 2\alpha-\beta$. Consequently, in the region to the left of the slanting lines, the quarter-circles give no contribution. Therefore the integral in (47) vanishes, and $u(r,\theta,t) = 1$.

We have thus verified that in the "geometrical" region where neither diffraction nor reflection plays a role, the solution $u(r,\theta,t)$ given by (41) obeys the geometrical propagation laws, and is in fact the same as for the wedge solution of section 2. We have not been able to find a method of summing the series (41) between the reflected pulse and the cone. Probably numerical methods must be employed.

All the methods of this paper may be carried over to arbitrary angles of incidence. The solution in that case involves ϕ dependence, and the orders of the Legendre functions to be used in the expansion will be related to this ϕ dependence, since a typical term in the expansion to be employed will be $P^m_{\nu_m}(-\cos\theta)\cos m\phi$, where $P^m_{\nu_m}(-\cos\alpha)=0$. The integrals over k are identical with those already appearing here. Again it is

possible to show that the solution in the geometrical region is one or zero in the appropriate region, but the series for the diffraction region and for the reflection region prove intractable.

The range of applicability of the method of this paper may be stated roughly as for all problems in which the obstacle does not involve a fundamental length. If a length is involved, the integrals over k will usually be evaluable only in terms of series, rather than in closed form. The resulting multiple series will generally be intractable.

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